MTH 265. LN2B. Week 2 Wednesday Lecture Notes.
Area Corrected Approximations and More Integral Tests.

Example 0.1. Determine the convergence of $\sum_{n=1}^{\infty} \frac{2n-3}{n^2-3n+4}$ using the Integral Test.

Let $f_b(x) = \frac{2x-3}{x^2-3x+4}$; Check the conditions.

(i) x^2-3x+4 has no real roots since $\Delta = (-3)^2 - 4(1)(4) = 9-16 = -7 < 0$. So, $f_b(x)$ is continuous on $\mathbb{R}$.

(ii) $f_b'(x) = (x^2-3x+4)^{-2}\left[(x^2-3x+4)(2) - (2x-3)(2x-3)\right] = (x^2-3x+4)^{-2}\left[-2x^2+6x+1\right]$;

(iii) Do a sign chart on $f_b'(x)$. Since $f_b'(x)$ is continuous on $\mathbb{R}$, we only need to find the zeros of $f_b'(x)$.
Equivalently, find all $x \in \mathbb{R}$ such that $-2x^2+6x+1=0$;
By the quadratic formula: $x_{1,2} = \frac{1}{2(-2)}\left(-6 \pm \sqrt{(6)^2-4(-2)(1)}\right) = 0.177, 2.823$;

Chart:

$f_b(x)$	$\searrow$		$\nearrow$		$\searrow$
		0.177		2.823	
$f_b'(x)$	$(-)$		$(+)$		$(-)$
	Try $x=0$: $f_b'(0) = -0.0625$		Try $x=1$: $f_b'(1) = 0.75$		Try $x=3$: $f_b'(3) = -0.0625$

Therefore, $f_b(x)$ is decreasing for $x > 2.823$;

Therefore, we can apply the Integral Test on $\sum_{n=3}^{\infty}\left(\frac{2n-3}{n^2-3n+4}\right)$ with $f(x) = \frac{2x-3}{x^3-3x+4}$; Observe the change in starting index.

Then, $\int_3^{\infty} \frac{2x-3}{x^2-3x+4}\,dx = \lim_{b\to\infty} \int_3^b \frac{2x-3}{x^2-3x+4}\,dx \underset{du = 2x-3\,dx}{\overset{u = x^2-3x+4}{=}} \lim_{b\to\infty} \int_{x=3}^{x=b} \frac{1}{u}\,du = \lim_{b\to\infty}\left[\ln|u|\right]_{x=3}^{x=b}$

$= \lim_{b\to\infty}\left[\ln(b^2-3b+4) - \ln(2(3)-3)\right] = \lim_{b\to\infty} \ln(b^2-3b+4) - \ln(6) = \infty$, i.e. diverges.

By the Integral Test, $\sum_{n=3}^{\infty} \frac{2n-3}{n^2-3n+4}\,dx$ diverges. Therefore, $\sum_{n=1}^{\infty} \frac{2n-3}{n^2-3n+4}$ also diverges.

Definition 1. Area Corrected Approximation.

Let $\sum_{n=n_0}^{\infty} f(n)$ be a series identified to be convergent by the Integral Test on $f(x)$ with $x \in [n_0, \infty)$.
The N^{th} order area corrected approximation U_N of $\sum_{n=n_0}^{\infty} f(n)$ is defined as

$$U_N = S_N + \int_{N+1}^{\infty} f(x)\,dx \quad \text{with} \quad S_N = \sum_{k=n_0}^{N} f(k), \text{ the } N^{th} \text{ partial sum of } \sum_{n=n_0}^{\infty} f(n).$$

Proposition 1. Error of Area Corrected Approximations.

Let $S = \sum_{n=n_0}^{\infty} f(n)$ and let U_N be its N^{th} order area corrected approximation.

Then, the error $E_N = S - U_N$ satisfies $\boxed{0 < S - U_N < f(N+1)}$;

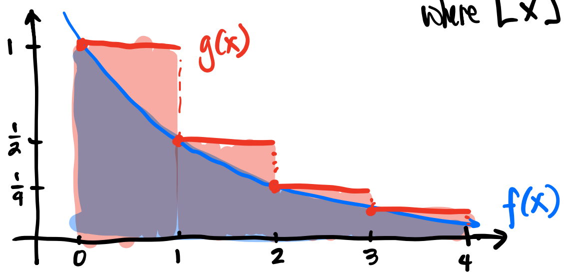
That is, U_N is always an overestimate and its error is bounded above by the 1^{st} term that is not included.

Justification for Proposition 1. (This is not a proof since details are missing. It's more of a proof sketch.)

Let $S = \sum_{n=n_0}^{\infty} f(n)$ be a series identified to be convergent by the Integral Test on $f(x)$ with $x \in [n_0, \infty)$.

Idea ①: The series $\sum_{n=n_0}^{\infty} f(n)$ can be described by the integral on $g(x) = f(\lfloor x \rfloor)$ on $x \in [n_0, \infty)$ where $\lfloor x \rfloor$ is the floor function.

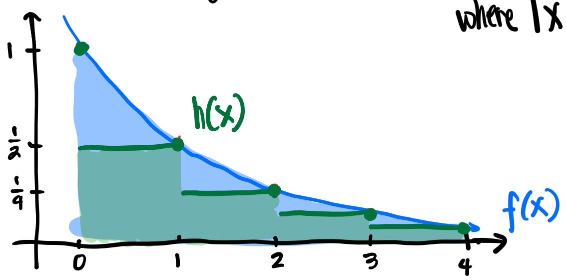
Illustrated for $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$:



Then, for all $n \in \mathbb{Z}_{\geq n_0}$: $\int_{n_0}^{\infty} f(x)\,dx < S$.

Idea ②: The series $\sum_{n=n_0}^{\infty} f(n)$ can also be described by $f(n_0) + \int_{n_0}^{\infty} h(x)$ with $h(x) = f(\lceil x \rceil)$ where $\lceil x \rceil$ is the ceiling function.

Illustrated for $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$:



Then, for all $n \in \mathbb{Z}_{\geq n_0}$: $S - f(n_0) < \int_{n_0}^{\infty} f(x)\,dx$;

Idea ③: Combining Ideas ① and ②: $\int_{n_0}^{\infty} f(x)\,dx < S < f(n_0) + \int_{n_0}^{\infty} f(x)\,dx$ (★)

Observe that $S - S_N = \sum_{n=N+1}^{\infty} f(n)$.

If we apply (★) with $n_0 = N+1$: $\int_{N+1}^{\infty} f(x)\,dx < S - S_N < f(N+1) + \int_{N+1}^{\infty} f(x)\,dx$;

Adding S_N to both sides: $S_N + \int_{N+1}^{\infty} f(x)\,dx < S < f(N+1) + S_N + \int_{N+1}^{\infty} f(x)\,dx$;

Recall that $U_N = S_N + \int_{N+1}^{\infty} f(x)\,dx$: $U_N < S < f(N+1) + U_N$;

Therefore, $0 < S - U_N = E_N < f(N+1)$ as desired.

Example 2.1. Consider $S = \sum_{n=1}^{\infty} \frac{1}{n^3}$.

(a) Determine its convergence using the Integral Test.

Let $f(x) = \frac{1}{x^3}$; Then, (i) $f(x)$ is continuous on $x \in \mathbb{R}$ with $x \neq 0$.

(ii) For $x \in [1, \infty)$: $f(x)$ is positive.

(iii) Since $f'(x) = (-1)\frac{1}{x^4}$ is always negative for $x \in [1, \infty)$, $f(x)$ is decreasing for $x \in [1, \infty)$.

We can apply the Integral Test on $\sum_{n=1}^{\infty} \frac{1}{n^3}$;

So, $\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_1^b x^{-3}\,dx = \lim_{b\to\infty}\left[\frac{1}{-3+1}x^{-3+1}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{2}b^{-2} + \frac{1}{2}(1)^{-2}\right] = \frac{1}{2} < \infty$.

By the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

(b) Find its 4th order area-corrected approximation U_4.

By defn, $U_4 = S_4 + \int_5^{\infty} \frac{1}{x^3}\,dx$; $S_4 = \sum_{n=1}^{4} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} = \frac{2035}{1728}$;

$\int_5^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2}(5)^{-2} = \frac{1}{50}$; $U_4 = \frac{2035}{1728} + \frac{1}{50} = \frac{51739}{43200} \approx 1.19766$;

(c) How accurate is U_4? Use Proposition 1.

From the proposition, $0 < E_4 = S - U_4 < f(4+1) = \frac{1}{5^3} = 0.008$; Therefore, $E_4 \in (0, 0.008)$.

(d) Using Proposition 1, find N minimal such that U_N is accurate to 4 decimal places.

From the proposition, $0 < E_4 = S - U_N < f(N+1)$;

We want to find N minimal such that $f(N+1) = (N+1)^{-3} < \frac{1}{2}(10^{-4}) = 0.00005$;

Then, $(N+1)^{-3} < \frac{1}{2}(10^{-4})$

$(N+1)^3 > 2(10^4)$ since both sides are positive.

$N+1 > (2(10^4))^{\frac{1}{3}} \approx 27.144$ since $g(x) = x^{\frac{1}{3}}$ is increasing on $x \in [1, \infty)$.

$N > 27.144 - 1 = 26.144$;

Choose $N = 27$;

! Alternatively, since $f(x) = x^{-3}$ is decreasing as required by the Integral Test, we can find the solution $x \in \mathbb{R}$ such that $f(x+1) = \frac{1}{2}(10^{-4})$ and choose $N \in \mathbb{Z}$ minimal such that $x < N$.

So, $(x+1)^{-3} = \frac{1}{2}(10^{-4})$; $x + 1 = [2(10^4)]^{\frac{1}{3}} \approx 27.144$; $x = 26.144$; Choose $N = 27$;

Example 2.2. Consider $\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$;

(a) Determine its convergence using the Integral Test.

Let $f(x) = \frac{\arctan(x)}{1+x^2}$;

(i) $1 + x^2$ has no zeroes and $\arctan(x)$ is continuous on $\mathbb{R}$. $\therefore f(x)$ is continuous on $\mathbb{R}$.

(ii) For $x \geq 0$, $\arctan(x) \in [0, \frac{\pi}{2})$. For all $x \in \mathbb{R}$, $1+x^2$ is positive. $\therefore f(x)$ is positive on $[0,\infty)$.

(iii) $f'(x) = (1+x^2)^{-2}\left[(1+x^2)(1+x^2)^{-1} - \arctan(x)(2x)\right] = (1+x^2)^{-2}\left[1 - 2x\arctan(x)\right]$;

Assuming $x \geq 1$, $\arctan(x) \geq \frac{\pi}{4} \approx 0.785$ since $\arctan(x)$ is an increasing function.

Then, $2x\arctan(x) \geq 2(1)\arctan(x) \geq 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2} = 1.507\ldots > 1$.

Therefore, $1 - 2x\arctan(x)$ is negative for $x \geq 1$.

Since $(1+x^2)$ is always positive, $f'(x)$ is negative for $x \in [1,\infty)$

$\therefore f(x)$ is decreasing on $[1,\infty)$.

We can apply the Integral Test on $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$;

Then, $\displaystyle\int_1^{\infty} \frac{\arctan(x)}{1+x^2}\,dx = \lim_{b\to\infty} \int_1^{\infty} \frac{\arctan(x)}{1+x^2}\,dx \underset{du=(1+x^2)^{-1}dx}{\overset{u=\arctan(x)}{=}} \lim_{b\to\infty} \int_{x=1}^{x=b} u\,du$

$$= \lim_{b\to\infty}\left[\frac{1}{2}u^2\right]_{x=1}^{x=b} = \frac{1}{2}\lim_{b\to\infty}\left[(\arctan(b))^2 - (\arctan(1))^2\right]$$

$$= \frac{1}{2}\left[\left(\lim_{b\to\infty}\arctan(b)\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \frac{\pi^2}{16}\right] < \infty.$$

By the Integral Test, $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$ converges!

(b) Using a calculator, find the N^{th} order area corrected approximation of S accurate to 3 decimal places.

Find N such that $E_N < f(N+1) < \frac{1}{2}(10^{-3})$;

Since $f(x)$ is decreasing, we can find $x \in \mathbb{R}$ such that $f(x+1) = \dfrac{\arctan(x+1)}{(x+1)^2+1} = \frac{1}{2}(10^{-3})$;

By Wolfram Alpha, $x \approx 54.7199$; Choose $N = 55$;

Then, $\displaystyle U_{55} = \sum_{n=1}^{55} \frac{\arctan(n)}{1+n^2} + \int_{56}^{\infty} \frac{\arctan(x)}{1+x^2}\,dx$

$$= \sum_{n=1}^{55} \frac{\arctan(n)}{1+n^2} + \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - (\arctan(56))^2\right] \overset{\text{Wolfram}}{\approx} 1.13533;$$

ANS: $\boxed{S \approx 1.135}$ is accurate to 3 decimal places.